The Uniqueness Structure of Simple Latent Trait Models*

HANS IRTEL

University of Regensburg

Abstract

A latent trait system is a set of subjects A, a set of items X, and a response function r mapping $A \times X$ into the real numbers. Numerical representations of such a system map A and X into the reals, such that r is represented by a numerical operation. It is shown for an additive latent trait system that its internal structure may be characterized by its automorphism group and that homogeneity and uniqueness of this group make the system ratio scalable. A non-additive case is also considered. Here the two factors are combined in a non-additive way but the system's internal structure induces an independent system on one of the two factors which is interval scalable.

Uniqueness Problems in Latent Trait Systems

Latent trait models have been introduced by Rasch (1960) and Birnbaum (1968) in order to improve the theoretical foundations of psychometric measurement. A consequence of this is that these models may be analyzed by measurement theoretic methods for specifying rigorously a set of sufficient conditions for the existence of the parameters and their uniqueness properties. Unfortunately there are three types of uniqueness problems involved which have been confused in the past. The first one is based on the fact that most measurement models do not define the homomorphisms which map the empirical structure they deal with into a given numerical structure uniquely. This is what is usually called the *uniqueness problem* in measurement theory and its solution is the set of admissible transformations of a scale (Suppes & Zinnes, 1963; Krantz, Luce, Suppes, & Tversky, 1971). The second type of uniqueness question is how different but equally suitable numerical models for one and the same empirical structure are related. This question is of less importance to measurement theory, since the essential fact about any scale is that all suitable numerical representations are isomorphic and so are the corresponding sets of admissible transformations (Krantz et al., 1971; Colonius, 1979). A measurement theoretic analysis of latent trait models usually is based on response probabilities and this creates a third type of uniqueness problem which is related to the statistical problem of how to estimate the response probabilities and how the uniqueness of the estimates depends on properties of a finite set of response data. This problem has been dealt with by Rasch (1960), Andersen (1973), and Fischer (1981).

Statistical properties of a latent trait model, like the existence of sufficient statistics for its parameters affect the statistical parameter estimation problem, but do not affect its scale type. The scale type of a measurement model can be thought of as the limit

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of over-all uniqueness one can get with an ideal set of data, namely the response *probabilities*.

This paper deals only with the first type of uniqueness question. Its main point is to analyze the uniqueness of latent trait systems by using structural properties of the response probabilities and not referring to numerical representations. This avoids any confusion with the second type of uniqueness problem mentioned earlier. In fact, several authors have confused the lack of uniqueness of homomorphisms into a fixed numerical relational system with isomorphic mappings between different numerical relational systems (Colonius 1979; Wottawa 1980; Roskam 1983) although others have made rather clear that the second type of problem does neither affect the scale type nor the set of meaningful statements which are possible with a given scale (Krantz et al. 1971).

Statistical estimation problems are not addressed here, which means that we start our analysis with a response function r which assigns a real valued number $r(a, x)$ to every pair (a, x) of subject and item. The response function is assumed to be fixed and derived from response probabilities, which do not have any nontrivial admissible transformations.

DEFINITION 1. Let A and X be nonempty sets and let r be a mapping from $A \times X$ into the set of real numbers Re. Then $\mathcal{L} = \langle A \times X, r \rangle$ is called a *latent trait system.*

We first consider only strictly additive latent trait systems. This means that the response function may be decomposed into independent components which are combined in an additive manner. These systems may be called *Rasch*-type since they essentially are equivalent to what has become known as the *Rasch*-model:

$$r(a, x) = \theta(a) - \epsilon(x). \tag{1}$$

The existence of such an additive decomposition imposes structural restrictions on the response function. These restrictions may be considered "empirical" properties of a latent trait system which, given the response function, are empirically testable (Hamerle & Tutz, 1980).

DEFINITION 2. A latent trait system $\mathcal{L} = \langle A \times X, r \rangle$ is called *strictly additive independent* iff for all a, b in A and all x, y in X

$$r(a, x) - r(b, x) = r(a, y) - r(b, y). \tag{2}$$

A latent trait system is additive decomposable in the sense of (1) iff it is strictly additive independent (Hamerle & Tutz, 1980). Latent trait theory does not use strictly additive representations of response probabilities directly, but uses mapping functions which restrict their range to the interval $[0, 1]$. An example is the so called *logistic* function $L(x) = 1/(1 + e^{-x})$ which has been used by Rasch (1960). The representation for response probabilities then is

$$p(a, x) = \frac{1}{1 + e^{-(\theta(a) - \epsilon(x))}}. \tag{3}$$

Since here the mapping function is specified explicitly and strictly monotone one may invert it and get

$$\ln \frac{p(a, x)}{1 - p(a, x)} = \theta(a) - \epsilon(x).$$

Transforming the response probabilities p by the inverse of the logistic creates a response function which has an additive representation in the sense of (1). It is important to note that the transformation also affects the empirical condition (2). In terms of response probabilities condition (2) then becomes

$$\frac{p(a,x)}{1-p(a,x)}\frac{1-p(b,x)}{p(b,x)} = \frac{p(a,y)}{1-p(a,y)}\frac{1-p(b,y)}{p(b,y)}.$$

Pfanzagl (1971), Hamerle (1979), and Fischer (1988) look at a representation which is more general than (1):

$$r(a,x) = H[\theta(a) - \epsilon(x)], \tag{4}$$

where H is some strictly monotone increasing function with appropriate domain. Pfanzagl (1971) and Hamerle (1979) derive θ, ϵ, and H from purely ordinal constraints on r which essentially form an additive conjoint measurement structure (Luce & Tukey, 1964). This means that only ordinal information in r is used and that any strictly monotone transformation of r results in an equivalent representation. However, since the type of ordinal constraints used in additive conjoint measurement ("double cancellation" or "quadruple condition") is not suitable as a restriction in statistical parameter estimation, these models are hard to apply in situations where response probabilities cannot be estimated by relative frequencies.

Clearly, if H is invertible, then (4) implies the following generalization of (2)

$$G\{H^{-1}[r(a,x)],\, H^{-1}[r(b,x)]\} = G\{H^{-1}[r(a,y)],\, H^{-1}[r(b,y)]\}, \tag{5}$$

with $G(x,y) = x - y$. Fischer (1988) derives (4) from (5) for a more general class of functions G. He also shows that if the response function values are stochastically independent response probabilities and if G is some function of the likelihood of a response event involving the subjects a, b and the item x, then H must be the logistic. This also shows that whithin the latent trait framework (4) becomes empirically applicable only if the function H is fixed and known. The reason simply is that for H unknown (5) neither is empirically testable, nor is it useful as a side condition in parameter estimation. An exception is $H(x) = c\,x + d$ (Suppes & Zinnes, 1963; Fischer, 1988) which makes (5) equivalent to (2), such that (4) and (1) are *data eqivalent* in the sense of Adams, Fagot, and Robinson (1965). The parameters c and d are not identifyable and thus may be set equal to 1 and 0 respectively without loss of generality.

In the following we ignore all problems related to statistical parameter estimation and only look at latent trait systems with fixed and known response functions. These are treated as the empirical basis for latent trait models in the same way as fundamental measurement theory treats "empirical" relational systems as empirical basis for measurement. With this restriction, the uniqueness properties that will be derived are properties that become effective only, if an actual set of data comes sufficiently close to those properties of latent trait systems that will be required. In this sense they describe the optimum degree of uniqueness one can get within a given latent trait system.

Intrinsic Uniqueness

The following presentation applies methods for characterizing the scale type of a measurement structure which have been developed by Narens (1985). These methods are not based on numerical representations but make it possible to characterize the scale type of a latent trait system by using only its intrinsic primitives.

Our analysis is based on a special type of *automorphisms* of a latent trait system $\mathcal{L} = \langle A{\times}X, r\rangle$, which generally are one-to-one mappings ζ from $A{\times}X$ onto itself such that $r[\zeta(a,x)] = r(a,x)$. This is similar to what Roberts and Rosenbaum (1988/89) call "tight value automorphisms" in the context of finite weak orderings with an additional binary relation ("valued digraphs"). This terminology is slightly different from the one used by Narens (1985). He looks at automorphisms which preserve relations like $r(a,x) \geq r(b,y)$ iff $r[\zeta(a,x)] \geq r[\zeta(b,y)]$. Since latent trait systems are based on real valued functions we have to use value preserving automorphisms. We simply use the term *automorphisms* but keep in mind that these mappings not only preserve relations but values. We use the term *order automorphism* for denoting the conventional automorphisms which preserve relations.

The following concept has been introduced by Luce and Cohen (1983). They look at automorphisms of conjoint structures, which strongly resemble latent trait systems. The difference is that conjoint structures have an empirical ordering relation as their basic primitive, while latent trait systems have a response function.

DEFINITION 3. Let $\mathcal{L} = \langle A \times X, r \rangle$ be a latent trait system. An automorphism ζ of $\mathcal{L}$ is *factorizable* iff there exist one-to-one functions α from A onto itself and ξ from X onto itself, such that for all a in A and all x in X $\zeta(a,x) = (\alpha(a), \xi(x))$. The functions α and ξ will be denoted *component transformations* induced by ζ.

We use $\mathcal{M}$ to denote the set of all factorizable automorphisms of a latent trait system $\mathcal{L}$ and $\mathcal{M}_A$ and $\mathcal{M}_X$ for the induced component transformations on A and X respectively.

LEMMA 1. *Let $\mathcal{L} = \langle A \times X, r \rangle$ be a latent trait system, $\mathcal{M}$ its set of factorizable automorphisms and $\mathcal{M}_A$ and $\mathcal{M}_X$ its respective sets of induced component transformations. Then with function composition $\mathcal{M}$, $\mathcal{M}_A$, and $\mathcal{M}_X$ are groups.*

Proof. The proof closely follows the proof of Lemma 1 of Luce and Cohen (1983). Let ι_A, ι_X, and $\iota = \langle \iota_A, \iota_X \rangle$ be the identity maps on A, X, and $A \times X$ respectively. Then these are the identities of the three groups. Let $\langle \alpha, \xi \rangle$ be in $\mathcal{M}$, then $\langle \alpha, \xi \rangle^{-1} = \langle \alpha^{-1}, \xi^{-1} \rangle$ also is in $\mathcal{M}$, since $r(\alpha^{-1}(a), \xi^{-1}(x)) \neq r(a,x)$ leads to the contradiction $r(\alpha^{-1}(\alpha(a)), \xi^{-1}(\xi(x))) \neq r(a,x)$. From associativity for function composition it follows that $\mathcal{M}$, $\mathcal{M}_A$, and $\mathcal{M}_X$ are groups. □

In order to characterize the structural properties of a latent trait system, we use its automorphism group. Two properties have turned out to be especially useful: homogeneity and uniqueness.

DEFINITION 4. Let $\mathcal{L} = \langle A \times X, r \rangle$ be a latent trait system. Its group $\mathcal{M}$ of factorizable automorphisms is 1/2-*point homogenous on* A iff for any a, b in A there exists a factorizable automorphism $\langle \alpha, \xi \rangle$ in $\mathcal{M}$ such that $\alpha(a) = b$. $\mathcal{M}$ is 1/2-*point unique on* A iff for all a in A and for all factorizable automorphisms $\langle \alpha, \xi \rangle$, $\langle \alpha', \xi' \rangle$ in $\mathcal{M}$ the condition $\alpha(a) = \alpha'(a)$ implies that $\langle \alpha, \xi \rangle = \langle \alpha', \xi' \rangle$. Analog definitions may be given for 1/2-*point homogenous on* X and 1/2-*point unique on* X. $\mathcal{L}$ will be called 1/2-*point homogenous* and 1/2-*point unique* iff the respective properties hold for both components simultanously.

In order to use automorphisms to characterize the structural properties of latent trait systems, we have to make sure that any of them exist. This means that we have to assume infinite sets of subjects and items, since our systems essentially are ordered (Def. 7). One way to construct an automorphism on a latent trait system is via inverting the response function.

DEFINITION 5. A latent trait system $\mathcal{L} = \langle A \times X, r \rangle$ is called *invertible* iff for every fixed x in X the function $f_x(a) = r(a, x)$ and for every fixed a in A the function $f_a(x) = r(a, x)$ is a one-to-one function from A and X respectively into Re such that for every s in $r(A \times X)$ the inverse $f_x^{-1}(s)$ is a unique element a in A and the inverse $f_a^{-1}(s)$ is a unique element x in X.

THEOREM 1. *Let $\mathcal{L} = \langle A \times X, r \rangle$ be a latent trait system and let $\mathcal{M}$ be its group of factorizable automorphisms. If $\mathcal{L}$ is invertible then $\mathcal{M}$ is $1/2$-point unique.*

Proof. Suppose both $\langle \alpha, \xi \rangle$ and $\langle \alpha', \xi' \rangle$ are in $\mathcal{M}$ and $\alpha(a) = \alpha'(a) = b$ for some a in A. Then $r(b, \xi(x)) = r(b, \xi'(x))$ and it follows from invertibility that $\xi = \xi'$. By invertibility in A we then get $\alpha = \alpha'$ and thus $\langle \alpha, \xi \rangle = \langle \alpha', \xi' \rangle$. This shows 1/2-point uniqueness on A. 1/2-point uniqueness on X may be shown in the same way. □

This theorem shows that if a latent trait system $\mathcal{L}$ is invertible then its set of factorizable automorphisms is 1/2-point unique. This result is closely related to Theorem 12 of Luce and Cohen (1983). They show that the group of factorizable order automorphisms of conjoint structures which are order independent and unrestrictedly solvable either satisfies 1- or 2-point uniqueness (Def. 8).

Theorem 1 also shows that for invertible systems we have to look at the homogeneity properties of the automorphism groups. Note that the *degree of homogeneity (uniquenes)* of a group of automorphisms is the largest (smallest) n, such that it is n-point homogenous (unique), given such a n exists. If such a n does not exist then the degree of homogeneity (uniqueness) is said to be ∞. Also, remember that for any group of automorphisms with a finite degree of uniqueness the degree of homogeneity is less than or equal to its degree of uniqueness (Narens, 1985).

Up to now we have not used condition (2). Going to additive systems means to use real addition as an operation for combining the two factors of a latent trait system. We therefore need a stronger version of invertibility in order to define the corresponding automorphisms in our system.

DEFINITION 6. A latent trait system $\mathcal{L} = \langle A \times X, r \rangle$ is called *A-solvable* iff for any fixed x in X and any real number s there exists a unique a in A such that $r(a, x) = s$. X-solvability is defined in an analogous way. $\mathcal{L}$ is *solvable* iff it is A- and X-solvable.

THEOREM 2. *Let $\mathcal{L} = \langle A \times X, r \rangle$ be a solvable latent trait system which is strictly additive independent and let $\mathcal{M}$ be its group of factorizable automorphisms. Then $\mathcal{M}$ is $1/2$-point homogenous and $1/2$-point unique.*

Proof. Let a' and b' be in A and define $\alpha(a') = b'$. Then by strict additive independence

$$k_\alpha = r(\alpha(a'), x) - r(a', x)$$

is independent of x. For all a in A define $\alpha(a)$ as the solution to

$$r(\alpha(a), x) - r(a, x) = k_\alpha$$

which again is independent of x. Then define ξ on X as the solution of

$$r(a, \xi(x)) - r(a, x) = -k_\alpha$$

for some a in A and note that by strict additive independence ξ is independent of a. Then clearly $\langle \alpha, \xi \rangle$ is factorizable. It also is an automorphism of $\mathcal{L}$ because by definition

$$\begin{aligned} r(\alpha(a), \xi(x)) &= r(\alpha(a), x) - k_\alpha \\ &= (r(a,x) + k_\alpha) - k_\alpha \\ &= r(a,x). \end{aligned}$$

This shows that $\mathcal{M}$ is 1/2-point homogenous on A. 1/2-point homogeneity on X may be shown in the same way. 1/2-point uniqueness of $\mathcal{L}$ follows from Theorem 1. □

The proof of Theorem 2 shows what the automorphisms of a strictly additive latent trait system look like. They have the form

$$\alpha(a) = f_x^{-1}(f_x(a) + k_\alpha).$$

An equivalent form has been derived by Colonius (1979) for showing that each member of the automorphism group of any *representation* of a strictly additive independent latent trait system of the form

$$r(a,x) = F[\theta(a), \xi(x)]$$

with F strictly increasing in the first and strictly decreasing in the second argument may be characterized by a single parameter. Theorem 2 leads to an equivalent conclusion, but does not refer to any representation at all. It thus shows that 1/2-point uniqueness and 1/2-point homogeneity are intrinsic properties of a strictly additive latent trait system and are not imposed on it by the selection of a special representation as has been claimed, among others, by Colonius (1979), Wottawa (1980), and Roskam (1983).

Component Structures

The Additive Case

Factorizable automorphisms are especially useful for looking at component structures of latent trait systems. Let $\mathcal{L} = \langle A \times X, r \rangle$ be a solvable latent trait system which satisfies strict additive independence and let $\mathcal{M}$ be its group of factorizable automorphisms. Choose x from X and define a function δ on $A \times A$ by

$$\delta(a,b) = r(a,x) - r(b,x). \tag{6}$$

Then δ is independent of x by strict additive independence and the set of component transformations $\mathcal{M}_A$ is the automorphism group of the system $\langle A, \delta \rangle$:

$$\begin{aligned} \delta(\alpha(a), \alpha(b)) &= r(\alpha(a), x) - r(\alpha(b), x) \\ &= r(\alpha(a), \xi(x)) - r(\alpha(b), \xi(x)) \\ &= r(a,x) - r(b,x) \\ &= \delta(a,b). \end{aligned}$$

We need some more definitions for looking at the homogeneity and uniqueness properties of component structures.

DEFINITION 7. A latent trait system $\mathcal{L} = \langle A \times X, r \rangle$ is called *ordinal independent in* A iff for all a, b in A and all x, y in X

$$r(a,x) \geq r(b,x) \text{ iff } r(a,y) \geq r(b,y). \tag{7}$$

Note that strict additive independence implies ordinal independence in A. Ordinal independence in A is sufficient to define a relation $\succeq_A$ on A by

$$a \succeq_A b \ \text{ iff } \ r(a, x) \geq r(b, x) \tag{8}$$

for some x in X, such that $\langle A, \succeq_A \rangle$ is a weak order.

Definition 8. Let $\mathcal{L} = \langle A, \succeq_A, r \rangle$ be a system, where A is a set, $\langle A, \succeq_A \rangle$ is a weak order, and r, for some finite k, is a mapping from A^k into Re, and let $\mathcal{M}$ be a subgroup of the automorphisms of $\mathcal{L}$. $\mathcal{M}$ satisfies *n-point homogeneity* iff for any $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in A, which satisfy $a_1 \succ_A \cdots \succ_A a_n$ and $b_1 \succ_A \cdots \succ_A b_n$, there exists an automorphism α in $\mathcal{M}$ such that $\alpha(a_i) = b_i$, for $i = 1, \ldots, n$. $\mathcal{M}$ satisfies *n-point uniqueness* iff for all α, β in $\mathcal{M}$ and $a_1, \ldots, a_n$ in A with $a_1 \succ_A \cdots \succ_A a_n$ the condition $\alpha(a_i) = \beta(a_i)$, for $i = 1, \ldots, n$ implies that $\alpha = \beta$.

It is easy to show that the automorphism group of the system $\langle A, \delta \rangle$ which has been defined earlier is 1-point homogenous and 1-point unique and thus $\langle A, \delta \rangle$ is ratio scalable. Since δ also is independent of X it allows specifically objective ratio scale comparisons of subjects (Irtel, 1987).

A Nonadditive Case

In this section we look at a latent trait system which preserves many of the properties of strict additivity but does this only for one of the two components. Such a system has been introduced by Birnbaum (1968). Its main difference to a strictly additive system is that invariant quantitative statements about subjects involve three of them. This is a common situation with interval scales where ratios of intervals are invariant under admissible transformations.

Definition 9. A latent trait system $\mathcal{L} = \langle A \times X, r \rangle$ is called *affine independent in A* iff for all a, b, c in A and all x, y in X whenever the expressions are defined

$$\frac{r(a, x) - r(b, x)}{r(c, x) - r(b, x)} = \frac{r(a, y) - r(b, y)}{r(c, y) - r(b, y)}. \tag{9}$$

Note that affine independence in A implies ordinal independence in A (7). Thus it also allows the definition of a weak order $\langle A, \succeq_A \rangle$ which is independent of X. So Def. 8 may be applied also.

Theorem 3. *Let $\mathcal{L} = \langle A \times X, r \rangle$ be a latent trait system, affine independent in A and A-solvable. Let $\mathcal{M}_A$ be the set of component transformations α on A which satisfy*

$$\frac{r(\alpha(a), x) - r(\alpha(b), x)}{r(\alpha(c), x) - r(\alpha(b), x)} = \frac{r(a, x) - r(b, x)}{r(c, x) - r(b, x)} \tag{10}$$

for all a, b, c in A and x in X, whenever the expressions are defined. Then, under function composition, $\mathcal{M}_A$ is a group, is 2-point homogenous, and 2-point unique.

Proof. Showing that $\mathcal{M}_A$ is a group is left to the reader. We show 2-point homogeneity. Let a, b, a', and b' be four elements of A with $a \succ_A b$ and $a' \succ_A b'$. Define $\alpha(a) = a'$ and $\alpha(b) = b'$. Then let k_α be defined by

$$k_\alpha \quad = \quad \frac{r(a', x) - r(b', x)}{r(a, x) - r(b, x)}.$$

Note that because of affine independence in A the constant k_α does not depend on x. Then extend α on all of A by defining $\alpha(c)$ for all c in A as the solution to

$$r(\alpha(c), x) = k_\alpha[r(c, x) - r(b, x)] + r(b', x).$$

We have to show that α is an element of $\mathcal{M}_A$. By definition we have

$$\frac{r(\alpha(c), x) - r(b', x)}{r(c, x) - r(b, x)} \quad = \quad k_\alpha \quad = \quad \frac{r(a', x) - r(b', x)}{r(a, x) - r(b, x)}.$$

This implies that α is in $\mathcal{M}_A$, since $a' = \alpha(a)$ and $b' = \alpha(b)$. Thus $\mathcal{M}_A$ is 2-point homogenous, since the choice of a, b, a', and b' was arbitrary. It remains to show that $\mathcal{M}_A$ is 2-point unique. Suppose we have two transformations of $\mathcal{M}_A$, α and β, which coincide at two points a and b. It immediately follows from (10) that α and β also coincide at any other point c in A: From

$$\begin{aligned}\frac{r(\alpha(a), x) - r(\alpha(b), x)}{r(\alpha(c), x) - r(\alpha(b)x)} &= \frac{r(a, x) - r(b, x)}{r(c, x) - r(b, x)} \\ &= \frac{r(\beta(a), x) - r(\beta(b), x)}{r(\beta(c), x) - r(\beta(b), x)} \\ &= \frac{r(\alpha(a), x) - r(\alpha(b), x)}{r(\beta(c), x) - r(\alpha(b), x)}\end{aligned}$$

we get $r(\alpha(c), x) = r(\beta(c), x)$ which implies $\alpha(c) = \beta(c)$ by A-solvability. □

Affine independence corresponds to the Birnbaum-model (Birnbaum, 1968). It induces a 3-argument function

$$\gamma(a, b, c) = \frac{r(a, x) - r(b, x)}{r(c, x) - r(b, x)} \tag{11}$$

on A which is independent of X. Theorem 3 shows that the automorphism group of the induced system $\langle A, \gamma\rangle$ is 2-point homogenous and 2-point unique. Thus the system is interval scalable.

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